

Asymptotically Friedmann self-similar scalar field solutions with potential

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We investigate self-similar solutions which are asymptotic to the Friedmann universe at spatial infinity and contain a scalar field with potential. The potential is required to be exponential by self-similarity. It is found that there are two distinct one-parameter families of asymptotic solutions, one is asymptotic to the proper Friedmann universe, while the other is asymptotic to the quasi-Friedmann universe, i.e., the Friedmann universe with anomalous solid angle. The asymptotically proper Friedmann solution is possible only if the universe is accelerated or the potential is negative. If the potential is positive, the density perturbation in the asymptotically proper Friedmann solution rapidly falls off at spatial infinity, while the mass perturbation is compensated. In the asymptotically quasi-Friedmann solution, the density perturbation falls off only in proportion to the inverse square of the areal radius and the relative mass perturbation approaches a nonzero constant at spatial infinity. The present result shows that a necessary condition holds in order that a self-gravitating body grows self-similarly due to the constant accretion of quintessence in an accelerating universe.

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I. INTRODUCTION

It is now widely believed that the expansion of our universe got accelerated in its early phase of evolution, which is called *inflation*. Under reasonable assumptions, this inflation implies that some form of matter fields with largely negative pressure may have dominated the energy of the universe. The simplest inflation models postulate that a scalar field with flat potential would have induced this early-phase acceleration. On the other hand, the independent observations of supernovae, the cosmic microwave background, and the large scale structure has recently revealed that our universe is currently in a phase of accelerated expansion [1, 2]. This implies that the energy of our universe is currently dominated by some form of matter fields with largely negative pressure. Such matter fields are termed as *dark energy*. Although we do not know at present what the dark energy is, there are many possible candidates proposed. The first and simplest model is a cosmological constant. Phenomenologically, the perfect fluid model with an equation of state $p = w\rho$ is often adopted from a data-analysis point of view, where w might be constant or time-dependent. On the other hand, the simplest model for varying dark energy from a physical point of view is again a scalar field with flat potential or possibly some other dynamical fields with appropriate potential [3, 4]. We here call such scalar field models for dark energy *quintessence*. There are many variants of these varying dark energy models.

If we restrict ourselves to the evolution of the homogeneous and isotropic universe, the perfect fluid and the quintessence models of dark energy play basically the same role with equivalent model functions, which is the equation of state in the former and the potential in the latter. However, once we turn our attention to inhomogeneities and/or anisotropy, these two classes of models may show significant differences. Moreover, the response to inhomogeneous perturbations may distinguish the models which are degenerate in the homogeneous and isotropic evolution. Hence, it is indispensable to study inhomogeneities to distinguish the dark matter models. Our main interest in this paper is in the interaction between dark energy and black holes.

The problem of mass accretion onto black holes in an expanding universe was raised by Zel'dovich and Novikov [5], where they argued that the black-hole mass could increase self-similarly in proportion to the cosmological time. Although their argument was based on Newtonian gravity, self-similar solutions also arise in general relativity due to the scale-free nature of the Einstein field equation. Self-similar solutions are essentially characterized by functions of $z \equiv r/t$ and can describe inhomogeneous dynamics. They are also physically relevant because they may describe the asymptotic behavior of more general solutions. This is called *self-similarity hypothesis* [6] and, in fact, this was shown to be the case in some spherically symmetric gravitational collapse [7]. See [8, 9] for a recent review of self-similar solutions and self-similarity hypothesis. See also [10] for a review of self-similar solutions in a more general context.

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As we have a static black-hole solution in the Minkowski background, which is static, it would be natural to expect that we may have a self-similar black hole in the power-law flat Friedmann background, which is self-similar. In the study of the growth of primordial black holes [11], Carr and Hawking [12] and subsequent authors [13, 14, 15, 16] found that, if we consider a perfect fluid with the equation of state $p = (\gamma - 1)\rho$ for $1 \leq \gamma \leq 2$, there are no self-similar solutions which have a black-hole event horizon and are asymptotic to the *proper* Friedmann solution at large distance, but there are self-similar solutions with a black-hole event horizon which are only asymptotic to the Friedmann universe with some remaining anomaly. It has been realized that all the latter solutions are only asymptotic to the Friedmann solution with anomaly in solid angle, which are termed as *asymptotically quasi-Friedmann solutions* [17, 18].

This historical problem has been recently revived by the discovery of the currently accelerated expansion of our universe. The accretion of dark energy or phantom energy onto a Schwarzschild black hole [19, 20] and a Schwarzschild-de Sitter black hole [22] has been discussed. The cosmological evolution partially taken into account, it was suggested that black holes may grow self-similarly due to the accretion of a scalar field with potential [21]. When the cosmological evolution is fully taken into account, however, it was shown [16] that there is *no* self-similar black-hole solution which is asymptotic to the *decelerated* Friedmann universe for a massless scalar field and a scalar field with positive potential. On the other hand, it has been recently found [23] that there *is* a one-parameter family of self-similar solutions which have a black-hole event horizon and are asymptotic to the proper *accelerated* Friedmann universe for a perfect fluid with $p = (\gamma - 1)\rho$ ($0 < \gamma < 2/3$). This strongly suggests that black holes can significantly grow due to the constant-rate accretion of dark energy in an accelerating universe. However, it should be noted that this phenomenological perfect fluid model for dark energy is ill-behaved in small-scale physics [18]. For a scalar field with such a flat potential that accelerates the Friedmann universe, it is still an open problem whether there is a self-similar black-hole solution which is asymptotic to the Friedmann universe. To answer this question, it is necessary to understand the properties of asymptotically Friedmann self-similar solutions containing a scalar field with potential, and this is investigated in the present paper. In spite of the motivation for self-similar black holes in the universe, the result obtained here generally applies to any objects which evolve in a self-similar manner and are embedded into the Friedmann universe containing a scalar field with potential.

This paper is organized as follows. In Sec. II, we present a general formulation for self-similar solutions containing a scalar field with potential. In Sec. III, we rewrite the field equations for nonlinear perturbations from the Friedmann solution. In Sec. IV, we find two independent one-parameter families of asymptotic solutions which are asymptotic to the Friedmann universe in different ways. In Sec. V, we present the physical properties of these asymptotic solutions. In Sec. VI, we summarize the paper. We use the units, in which $c = 1$.

II. SELF-SIMILAR SOLUTIONS WITH A SCALAR FIELD

We consider a single scalar field φ with potential $V(\varphi)$ as a matter field, whose stress-energy tensor is given by

$$T_{ab} = \varphi_{,a}\varphi_{,b} - g_{ab} \left(\frac{1}{2}\varphi_{,c}\varphi^{,c} + V(\varphi) \right). \quad (2.1)$$

As we will see later, this can accelerate the expansion of the Friedmann universe. We adopt general relativity as a theory of gravity. The Einstein equation for this system is given by

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa^2 \left[\varphi_{,a}\varphi_{,b} - g_{ab} \left(\frac{1}{2}\varphi_{,c}\varphi^{,c} + V(\varphi) \right) \right], \quad (2.2)$$

where $\kappa \equiv \sqrt{8\pi G}$ and the comma denotes the partial derivative. The equation of motion for the scalar field is given by

$$\square\varphi = \frac{dV(\varphi)}{d\varphi}, \quad (2.3)$$

where \square denotes the d'Alembertian associated with g_{ab} . We consider a spherically symmetric spacetime, in which the line element is given by

$$ds^2 = -e^{2\Phi(t,r)}dt^2 + e^{2\Psi(t,r)}dr^2 + R^2(t,r)d\Omega^2, \quad (2.4)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the unit sphere and the domain of θ and ϕ are $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

We assume that the spacetime is self-similar, which is defined by the existence of a vector field ξ^a such that

$$\mathcal{L}_\xi g_{ab} = 2g_{ab}, \quad (2.5)$$

where \mathcal{L}_ξ denotes the Lie derivative along ξ^a . This vector field ξ^a is called a homothetic Killing vector. If ξ^a is tilted to $(\partial/\partial t)^a$, nondimensional metric functions depend only on $z \equiv r/t$ [24], i.e.,

$$\Phi = \Phi(z), \quad \Psi = \Psi(z), \quad R = rS(z). \quad (2.6)$$

Then, the scalar field φ and its potential $V(\varphi)$ are of the following form [25]:

$$\varphi = \frac{2}{\kappa\lambda} \ln r + f(z), \quad (2.7)$$

$$V(\varphi) = V_0 e^{-\kappa\lambda\varphi}, \quad (2.8)$$

where V_0 and λ are constants. It should be noted that for simplicity we have assumed that r and t are positive. In fact, we can recover the results for the general case simply by replacing r , t and z with $|r|$, $|t|$ and $|z|$, respectively.

Since we are interested in self-similar solutions perturbed from the Friedmann universe at large distances, we can assume that the gradient of the scalar field is timelike near spacelike infinity. In such a case, we can take time slicing so that φ depends only on the time coordinate t , which we call the constant scalar field slicing. In fact, this coordinate system is equivalent to the comoving coordinates where only the diagonal components of the stress-energy tensor are nonvanishing. In the following we choose this slicing, so that we have

$$f(z) = -\frac{2}{\kappa\lambda} \ln z + \varphi_0 \quad (2.9)$$

and

$$\varphi = \frac{2}{\kappa\lambda} \ln t + \varphi_0, \quad (2.10)$$

where φ_0 is a constant. If the scalar field is massless, i.e., $V_0 = 0$, we can simply delete φ_0 because only the gradient of φ appears in the action. If the scalar field has a potential, we can renormalize the constant φ_0 in Eq. (2.10) into the factor V_0 in the scalar field potential by replacing V_0 with \tilde{V}_0 such that

$$V_0 e^{-\kappa\lambda\varphi_0} = \tilde{V}_0. \quad (2.11)$$

Therefore, we set $\varphi_0 = 0$ in the following.

In this coordinate system, tt , tr , rr , and $\theta\theta$ components of the Einstein equation, respectively, yield

$$\left\{ 2\frac{S''}{S} + 2\frac{S'}{S} + \left(1 + \frac{S'}{S}\right)^2 - 2\Psi' \left(1 + \frac{S'}{S}\right) \right\} - V_z^2 \left\{ 2\Psi' \frac{S'}{S} + \left(\frac{S'}{S}\right)^2 \right\} - \frac{e^{2\Psi}}{S^2} = -\kappa^2 \left[\frac{2}{\lambda^2 \kappa^2} V_z^2 + z^2 e^{2\Psi} V_0 \right], \quad (2.12)$$

$$\Phi' \frac{S'}{S} + \Psi' \left(1 + \frac{S'}{S}\right) - \frac{S''}{S} - \frac{S'}{S} = 0, \quad (2.13)$$

$$V_z^2 \left\{ 2\frac{S''}{S} + 2\frac{S'}{S} + \left(\frac{S'}{S}\right)^2 - 2\Phi' \frac{S'}{S} \right\} - \left\{ 2\Phi' \left(1 + \frac{S'}{S}\right) + \left(1 + \frac{S'}{S}\right)^2 \right\} + \frac{e^{2\Psi}}{S^2} = -\kappa^2 \left[\frac{2}{\lambda^2 \kappa^2} V_z^2 - z^2 e^{2\Psi} V_0 \right], \quad (2.14)$$

$$\begin{aligned} V_z^2 \left\{ \frac{S''}{S} + \frac{S'}{S} + \Psi'' + \Psi' + \Psi'^2 + (\Psi' - \Phi') \frac{S'}{S} - \Phi' \Psi' \right\} - \left\{ \frac{S''}{S} + \frac{S'}{S} + \Phi'' - \Phi' + \Phi'^2 + (\Phi' - \Psi') \left(1 + \frac{S'}{S}\right) - \Phi' \Psi' \right\} \\ = -\kappa^2 \left[\frac{2}{\lambda^2 \kappa^2} V_z^2 - z^2 e^{2\Psi} V_0 \right], \end{aligned} \quad (2.15)$$

where the prime denotes the ordinary derivative with respect to $\ln z$ and

$$V_z \equiv z e^{\Psi - \Phi} \quad (2.16)$$

is the relative velocity between the constant z surface to the constant r surface. The equation of motion for the scalar field becomes

$$-\frac{2}{\kappa\lambda} V_z^2 \left(\Phi' - \Psi' - 2\frac{S'}{S} - 1 \right) + \kappa\lambda V_0 z^2 e^{2\Psi} = 0. \quad (2.17)$$

Four of the five equations (2.12)–(2.15) and (2.17) are independent.

We derive the following two relations for later use. Adding Eq. (2.12) to Eq. (2.14) and using Eq. (2.13), we get

$$V_z^2 \Psi' - \Phi' = -\frac{2}{\lambda^2} V_z^2. \quad (2.18)$$

Subtracting Eq. (2.12) from Eq. (2.14) and using Eq. (2.13), we get

$$V_z^2 \left\{ \Psi' \left(1 + 2 \frac{S'}{S} \right) + \left(\frac{S'}{S} \right)^2 \right\} - \left\{ \Phi' \left(1 + 2 \frac{S'}{S} \right) + \left(1 + \frac{S'}{S} \right)^2 \right\} + \frac{e^{2\Psi}}{S^2} = \kappa^2 V_0 z^2 e^{2\Psi}. \quad (2.19)$$

III. NONLINEAR PERTURBATION FROM THE FRIEDMANN SOLUTION

A. The flat Friedmann solution in self-similar coordinates

The flat Friedmann spacetime is given by the following line element:

$$ds^2 = -dt^2 + a(t)^2 (d\bar{r}^2 + \bar{r}^2 d\Omega^2). \quad (3.1)$$

The scale factor $a(t)$ satisfies the Friedmann equation

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \kappa^2 \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right), \quad (3.2)$$

where the dot denotes the derivative with respect to t . $\varphi = \varphi(t)$ satisfies the equation of motion

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \quad (3.3)$$

For these equations, we have a power-law solution

$$a = a_0 t^\alpha, \quad (3.4)$$

$$\varphi = \frac{2}{\kappa\lambda} \ln t, \quad (3.5)$$

where a_0 is a constant and α and V_0 are given by

$$\alpha = \frac{2}{\lambda^2}, \quad (3.6)$$

$$V_0 = \frac{2(6 - \lambda^2)}{\kappa^2 \lambda^4}. \quad (3.7)$$

This is obviously compatible with Eq. (2.10). A massless scalar field formally corresponds to $\lambda^2 = 6$, where $\alpha = 1/3$ and the cosmic expansion is decelerated. For a nontrivial potential, if $0 < \lambda^2 < 2$, then the cosmic expansion is accelerated, while, if $\lambda^2 > 2$, then the cosmic expansion is decelerated. For $\lambda^2 > 6$, the potential becomes negative.

If $\alpha \neq 1$, relating r to \bar{r} through

$$a_0 \bar{r} = \frac{r^{1-\alpha}}{|1-\alpha|}, \quad (3.8)$$

we can rewrite the flat Friedmann solution in the standard form for self-similar spacetimes, where

$$e^\Phi = 1, \quad e^\Psi = z^{-\alpha}, \quad S = \frac{1}{|1-\alpha|} z^{-\alpha}. \quad (3.9)$$

So, the power-law flat Friedmann solution is self-similar. If $\alpha = 1$, the flat Friedmann solution is still self-similar but the homothetic Killing vector is parallel to $(\partial/\partial t)^a$ (see e.g. [26]). This case needs a special treatment and we do not consider this case in the present paper. It should be emphasized that from Eq. (3.8) $\bar{r} \rightarrow 0$ and $\bar{r} \rightarrow \infty$ correspond to $r \rightarrow 0$ and $r \rightarrow \infty$, respectively, for $0 < \alpha < 1$, while this is reversed for $\alpha > 1$. When we study spatial infinity in general case, we should take the limit $z^{1-\alpha} \rightarrow \infty$ for fixed t [27].

B. Field equations for nonlinear perturbation

Since we are interested in self-similar solutions which are asymptotic to the Friedmann solution, we write general spherically symmetric self-similar solutions in the following form:

$$e^\Phi = e^{A(z)}, \quad e^\Psi = z^{-\alpha} e^{B(z)}, \quad S = \frac{1}{|1-\alpha|} z^{-\alpha} e^{C(z)}, \quad \varphi = \frac{2}{\kappa\lambda} \ln t + D(z). \quad (3.10)$$

As for the gradient of the scalar field, we get

$$\varphi_{,a} \varphi^{,a} = \frac{1}{t^2} \left[-e^{-2A} \left(\frac{2}{\kappa\lambda} - D' \right)^2 + z^{2\alpha-2} e^{-2B} D'^2 \right]. \quad (3.11)$$

The above equation implies that, if A and B are finite and D' is sufficiently small, we can choose the constant scalar field slicing, where D is a constant D_0 . This is the case where the solution is asymptotic to the flat Friedmann solution. Hereafter we take the constant scalar field slicing. We can set $D_0 = 0$. Then we have

$$e^\Phi = e^{A(z)}, \quad e^\Psi = z^{-\alpha} e^{B(z)}, \quad S = \frac{1}{|1-\alpha|} z^{-\alpha} e^{C(z)}, \quad \varphi = \frac{2}{\kappa\lambda} \ln t. \quad (3.12)$$

Substituting the above, we derive the following set of ordinary differential equations for A , B and C' :

$$-\alpha A' + A' C' + (1-\alpha) B' + B' C' - C'' - (1-\alpha) C' - C'^2 = 0 \quad (3.13)$$

from Eq. (2.13),

$$A' - B' - 2C' - (3\alpha - 1)(e^{2A} - 1) = 0 \quad (3.14)$$

from Eq. (2.17),

$$V_z^2 B' - A' = 0 \quad (3.15)$$

from Eq. (2.18), and the constraint equation

$$(V_z^2 - 1)(C'^2 - 2\alpha C' - \alpha A') - 2C' + (\alpha - 1)^2 (e^{2B-2C} - 1) = 0 \quad (3.16)$$

from Eqs. (2.19), (3.14) and (3.15), where V_z^2 is given by

$$V_z^2 = z^{2-2\alpha} e^{2B-2A} \quad (3.17)$$

from Eq. (2.16).

IV. ASYMPTOTICALLY FRIEDMANN SOLUTIONS

Although we are most interested in asymptotically proper Friedmann solutions, we also study more general solutions to which the asymptotic scheme applies. So we only require that all A , B and C have finite limit values, i.e.,

$$A \rightarrow A_0, \quad B \rightarrow B_0, \quad C \rightarrow C_0 \quad (4.1)$$

at spatial infinity, i.e., as $z^{1-\alpha} \rightarrow \infty$. Then, A' , B' , and C' tend to vanish from l'Hospital's rule. Hereafter, we always choose $A_0 = B_0 = 0$ by rescaling the coordinates t and r , whereas C_0 may not vanish.

It is not so trivial how such asymptotic solutions are expanded around $z^{1-\alpha} \rightarrow \infty$. First, we note that only V_z^{-2} is explicitly of higher order in Eqs. (3.13)–(3.15) in the present limit. Equation (3.15) then implies that B' is always of higher order than A' . If we linearize Eqs. (3.13) and (3.14), we get

$$-\alpha A' - C'' - (1-\alpha) C' = 0, \quad (4.2)$$

and

$$A' - 2C' - 2(3\alpha - 1)A = 0, \quad (4.3)$$

respectively. Then, eliminating C' and C'' , we get

$$A'' - (5\alpha - 3)A' + 2(\alpha - 1)(3\alpha - 1)A = 0. \quad (4.4)$$

A general solution of the above equation is given by the linear combination of the following two independent solutions:

$$A = z^{3\alpha-1} \quad (4.5)$$

and

$$A = z^{2\alpha-2}. \quad (4.6)$$

The second solution is always valid in the limit $z^{1-\alpha} \rightarrow \infty$, while the first is valid only for $(3\alpha - 1)/(1 - \alpha) < 0$. Since α is positive, the first solution is valid only for $0 < \alpha < 1/3$ or $\alpha > 1$.

A. Asymptotically proper Friedmann solutions

To get the full form of the first solution (4.5), we use Eqs. (3.13)–(3.15) and the result is the following:

$$A = A_1 z^{3\alpha-1}, \quad (4.7)$$

$$B = B_1 z^{5\alpha-3}, \quad (4.8)$$

$$C = C_0 + C_1 z^{3\alpha-1} \quad (4.9)$$

in linear order, where

$$B_1 = \frac{3\alpha - 1}{5\alpha - 3} A_1, \quad (4.10)$$

$$C_1 = -\frac{1}{2} A_1. \quad (4.11)$$

It turns out that we need higher order terms to see whether we can have nontrivial solutions and whether C_0 vanishes or not in Eq. (3.16). It is cumbersome but straightforward to get higher order terms from Eq. (3.13)–(3.15). The result is

$$A = A_1 z^{3\alpha-1} + A_2 z^{5\alpha-3} + A_3 z^{6\alpha-2} + \dots, \quad (4.12)$$

$$B = B_1 z^{5\alpha-3}, \quad (4.13)$$

$$C = C_0 + C_1 z^{3\alpha-1} + C_2 z^{5\alpha-3} + C_3 z^{6\alpha-2} + \dots, \quad (4.14)$$

where the coefficients are all parametrized by A_1 as follows:

$$A_2 = \frac{\alpha}{\alpha - 1} A_1, \quad (4.15)$$

$$C_2 = \frac{4\alpha^2 - 3\alpha + 1}{2(5\alpha - 3)(1 - \alpha)} A_1, \quad (4.16)$$

$$A_3 = \frac{11\alpha - 1}{8\alpha} A_1^2, \quad (4.17)$$

$$C_3 = -\frac{1}{2} A_1^2. \quad (4.18)$$

For $\alpha > 1$, we consider the limit $z \rightarrow 0$. In this case, the lowest order of Eq. (3.16), which is of order z^0 , just yields $C_0 = 0$. Therefore, we get self-similar solutions which are asymptotic to the proper Friedmann solution with vanishing C_0 . These solutions are termed as asymptotically proper Friedmann solutions.

For $0 < \alpha < 1/3$, the situation is more complicated. In this case, we consider the limit $z \rightarrow \infty$. Then, the terms of order $z^{5\alpha-3}$ in Eqs. (4.12) and (4.14) get higher than those of order $z^{6\alpha-2}$. Substituting Eqs. (4.12)–(4.14) into Eq. (3.16), and using Eqs. (4.17) and (4.18), we can see that the terms of orders $z^{1+\alpha}$ and $z^{4\alpha}$ all cancel out. Also in this case, the nontrivial lowest order, which is of order z^0 , yields $C_0 = 0$. So, these self-similar solutions are asymptotically proper Friedmann solutions.

For $\alpha = 1/3$, from the linear order analysis, we find that $A = A_1$ and $C = C_1$ are constants, while B vanishes from Eq. (4.10). It should be noted that we may have higher order terms. As we have set $A_0 = 0$, we can set $A_1 = 0$ by

rescaling the time coordinate. Then, if we have higher order terms, they must satisfy Eq. (4.4) and this again yields $A = \text{const}$ and $A = z^{2\alpha-2}$. The latter case must be included into the next case. Hence, we can concentrate on the solution where $A = 0$, $B = 0$ and $C = C_1$. In this case, we can show $C_1 = 0$ from Eq. (3.16). Therefore, the solution coincides with the exact Friedmann solution.

In summary, there is a one-parameter family of asymptotically proper Friedmann self-similar solutions for $0 < \alpha < 1/3$ or $1 < \alpha$. There is no nontrivial asymptotically proper Friedmann self-similar solution for $1/3 \leq \alpha < 1$.

B. Asymptotically quasi-Friedmann solutions

Up to the nontrivial lowest order, the second solution (4.6) is given by

$$\begin{aligned} A &= A_1 z^{2\alpha-2}, \\ B &= B_1 z^{4\alpha-4}, \\ C &= C_0 + C_1 z^{2\alpha-2}, \end{aligned} \quad (4.19)$$

where

$$\begin{aligned} B_1 &= A_1, \\ C_1 &= \frac{\alpha}{1-\alpha} A_1. \end{aligned} \quad (4.20)$$

From the lowest order of Eq. (3.16), which is of order z^0 , we get

$$A_1 = \frac{(\alpha-1)^2}{2\alpha(\alpha+1)} (1 - e^{-2C_0}). \quad (4.21)$$

Higher order terms are expanded in terms of integer powers of $z^{2-2\alpha}$ and the coefficients are written by integer power of A_1 . Hence, if $C_0 = 0$, we have $A_1 = B_1 = C_1 = 0$ and the solution becomes trivial. Only if $C_0 \neq 0$, we have a nontrivial solution. We term these nontrivial solutions as asymptotically quasi-Friedmann solutions. There is a one-parameter family of such solutions for $0 < \alpha < 1$ or $1 < \alpha$.

V. PHYSICAL PROPERTIES OF THE SOLUTIONS

We have shown that there are two types of asymptotically Friedmann solutions with trivial and nontrivial asymptotic values for C . The first is asymptotically proper Friedmann and the second is asymptotically quasi-Friedmann. In this section we see their physical properties.

A. Solid angle anomaly

We have the following asymptotic form of the metric near spatial infinity:

$$ds^2 = -dt^2 + z^{-2\alpha} dr^2 + \frac{z^{-2\alpha}}{(1-\alpha)^2} e^{2C_0} r^2 d\Omega^2 \quad (5.1)$$

$$= -dt^2 + a_0^2 t^{2\alpha} (d\bar{r}^2 + e^{2C_0} \bar{r}^2 d\bar{\Omega}^2), \quad (5.2)$$

where $C_0 = 0$ and $C_0 \neq 0$ hold for asymptotically proper Friedmann and quasi-Friedmann solutions, respectively. If we consider a two-sphere on which $t = \text{const}$ and $r = \text{const}$, its area is given by $4\pi a_0^2 t^{2\alpha} \bar{r}^2 e^{2C_0}$, while the proper length of the radius on the constant t hypersurface is equal to $a_0 t^\alpha \bar{r}$. So the ratio of the area to the squared radius is not 4π but $4\pi e^{2C_0}$. Therefore, there is a surplus in the solid angle for $C_0 > 0$ and a deficit for $C_0 < 0$. Only for $C_0 = 0$, we have no anomaly in the solid angle.

We can see this metric in another way. When we consider the $\theta = \pi/2$ section, we get the line element

$$ds^2 = -dt^2 + a_0^2 t^{2\alpha} (d\bar{r}^2 + e^{2C_0} \bar{r}^2 d\phi^2) \quad (5.3)$$

$$= -dt^2 + a_0^2 t^{2\alpha} (d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2), \quad (5.4)$$

where $\bar{\phi} = e^{C_0} \phi$ and hence

$$0 \leq \bar{\phi} < 2\pi e^{C_0}. \quad (5.5)$$

Although the above line element is the same as that for the $\theta = \pi/2$ section of the Friedmann solution, the domain of the azimuthal angle is anomalous. In fact, there is a surplus in the azimuthal angle for $C_0 > 0$ and a deficit for $C_0 < 0$.

This kind of anomaly in the solid angle is already discussed in the context of static global monopoles and termed as solid angle deficit for $C_0 < 0$ [28]. Hence, we can say that the asymptotically quasi-Friedmann solutions are with solid angle surplus or deficit, while the asymptotically proper Friedmann solutions are not. It should be noted that despite the apparent similarity with conical singularities in cylindrically symmetric spacetimes, the spacetime with the solid angle anomaly is not flat even locally.

B. Density perturbation

It is also interesting to get insight into the difference of the two classes of asymptotic solutions in the density field at spatial infinity on the constant t hypersurface. The energy density ρ observed by a comoving observer is given by

$$\rho \equiv n_a n_b T^{ab} = \frac{1}{t^2} \left[\frac{1}{2} e^{-2A} \left(\frac{2}{\kappa \lambda} \right)^2 + \frac{2(6 - \lambda^2)}{\lambda^4 \kappa^2} \right] \simeq \frac{12}{\kappa^2 \lambda^4 t^2} \left[1 - \frac{\lambda^2}{3} A \right], \quad (5.6)$$

where n^a is a unit vector normal to the constant scalar field hypersurface. Hence, the background Friedmann density ρ_b , the density perturbation $\delta\rho$, and the density contrast Δ_ρ are, respectively, given by

$$\rho_b = \frac{12}{\kappa^2 \lambda^4 t^2}, \quad (5.7)$$

$$\delta\rho \equiv \rho(t, r) - \rho_b(t, r) \approx -\frac{4}{\kappa^2 \lambda^2 t^2} A, \quad (5.8)$$

$$\Delta_\rho \equiv \frac{\delta\rho}{\rho_b} \approx -\frac{\lambda^2}{3} A, \quad (5.9)$$

where the suffix b denotes quantities for the background Friedmann solution and the weak equality “ \approx ” denotes that the ratio of both sides approaches unity in the relevant limit. The asymptotic form of the physical areal radius R is given by

$$R = rS \approx \frac{1}{|1 - \alpha|} r z^{-\alpha} e^{C_0} \quad (5.10)$$

for both cases. So, the fall-off of the density perturbation in terms of the physical areal radius is given by

$$\delta\rho \propto -A_1 t^{-2} z^{3\alpha-1} \propto -A_1 \left(\frac{R_b}{t} \right)^{-\frac{3\alpha-1}{\alpha-1}} t^{-2} \quad (5.11)$$

and

$$\delta\rho \propto -A_1 t^{-2} z^{2\alpha-2} \propto -A_1 R_b^{-2} \quad (5.12)$$

for asymptotically proper Friedmann and quasi-Friedmann solutions, respectively. Therefore, the density perturbation rapidly falls off for asymptotically proper Friedmann solutions for the accelerated case $\alpha > 1$. It falls off as R_b^{-2} for asymptotically quasi-Friedmann solutions. For asymptotically proper Friedmann solutions with $0 < \alpha < 1/3$, where the potential is negative, the fall-off is as slow as $R_b^{-(1-3\alpha)/(1-\alpha)}$ on the constant t slice, which is much slower than for asymptotically quasi-Friedmann solutions.

C. Mass perturbation

In spherically symmetric spacetimes, the Misner-Sharp mass m is known to be a well-behaved quasilocal mass defined as [29, 30]

$$\frac{2m}{R} \equiv 1 + e^{-2\Phi} R_{,t}^2 - e^{-2\Psi} R_{,r}^2. \quad (5.13)$$

In the present formulation, this can be rewritten as

$$\frac{M}{S} = 1 + \frac{1}{(1-\alpha)^2} z^{2-2\alpha} e^{2C-2A} (-\alpha + C')^2 - \left(1 + \frac{C'}{1-\alpha}\right)^2 e^{2C-2B}, \quad (5.14)$$

where M is the nondimensional mass defined by

$$M \equiv \frac{2m}{r}. \quad (5.15)$$

For the flat Friedmann solution, this quantity becomes

$$\left(\frac{2m}{R}\right)_b = \frac{\alpha^2}{(1-\alpha)^2} z^{2-2\alpha} = \alpha^2 \left(\frac{R_b}{t}\right)^2. \quad (5.16)$$

The perturbation for this quantity

$$\delta\left(\frac{2m}{R}\right) \equiv \left(\frac{2m}{R}\right)(z) - \left(\frac{2m}{R}\right)_b(z) \quad (5.17)$$

is given by

$$\delta\left(\frac{2m}{R}\right) \approx -\frac{\alpha}{(1-\alpha)^2} A_1 z^{1+\alpha} = -\alpha |1-\alpha|^{(3\alpha-1)/(1-\alpha)} A_1 \left(\frac{R_b}{t}\right)^{(1+\alpha)/(1-\alpha)} \quad (5.18)$$

for asymptotically proper Friedmann solutions and

$$\delta\left(\frac{2m}{R}\right) \approx \frac{\alpha^2}{(1-\alpha)^2} (e^{2C_0} - 1) z^{2-2\alpha} = \alpha^2 (e^{2C_0} - 1) \left(\frac{R_b}{t}\right)^2 \quad (5.19)$$

for asymptotically quasi-Friedmann solutions. Hence, the ratio of the perturbation to the background value

$$\Delta \equiv \delta\left(\frac{2m}{R}\right)(z) / \left(\frac{2m}{R}\right)_b(z) \quad (5.20)$$

is given by

$$\Delta \approx -\frac{1}{\alpha} A_1 z^{3\alpha-1} = -\frac{|1-\alpha|^{(3\alpha-1)/(1-\alpha)}}{\alpha} A_1 \left(\frac{R_b}{t}\right)^{(3\alpha-1)/(1-\alpha)} \quad (5.21)$$

and

$$\Delta \approx e^{2C_0} - 1 \quad (5.22)$$

for asymptotically proper Friedmann and quasi-Friedmann solutions, respectively.

It should be noted that since m and R are nonlinearly perturbed for asymptotically quasi-Friedmann solutions, this ratio depends on whether we compare the perturbation at the same r/t or at the same R/t . Noting that S is also perturbed, the mass perturbation $\delta M(z) \equiv M(z) - M_b(z)$ is directly given by

$$\Delta_M \equiv \frac{\delta M}{M_b} = (1 + \Delta) e^C - 1. \quad (5.23)$$

For asymptotically proper Friedmann solutions, Δ_M is calculated as

$$\Delta_M \approx -\frac{\alpha+2}{2\alpha} A_1 z^{3\alpha-1} \propto -A_1 \left(\frac{R_b}{t}\right)^{(3\alpha-1)/(1-\alpha)}. \quad (5.24)$$

Hence, the relative mass perturbation Δ_M tends to vanish at spatial infinity for both $\alpha > 1$ and $0 < \alpha < 1/3$. This also implies that the mass perturbation δm itself tends to vanish for $\alpha > 1$ but diverge for $0 < \alpha < 1/3$ as it is proportional to $(R_b/t)^{2/(1-\alpha)} t$. For asymptotically quasi-Friedmann solutions, Δ_M is calculated as

$$\Delta_M \approx e^{3C_0} - 1. \quad (5.25)$$

Hence, the relative mass perturbation tends to be constant. This is directly related to the solid angle anomaly. If Δ_M is positive (negative), there is a solid angle surplus (deficit). This situation is apparently opposite to the case of global monopoles, where the positive (negative) mass density implies deficit (surplus) in the solid angle. This is due to the fact that the Misner-Sharp mass is dominated by the first and third terms on the right-hand side of Eq. (5.13) for the static configuration, while it is by the second term, i.e., the kinematic term, for the flat Friedmann solution. The mass perturbation δm itself diverges as $(R/t)^3 t$ for both $0 < \alpha < 1$ and $1 < \alpha$.

So, in order to have an asymptotically proper Friedmann solution from an accelerated Friedmann universe, for which the potential is positive, we only need to perturb a finite amount of mass and the mass perturbation is compensated at spatial infinity. In contrast, in order to have an asymptotically quasi-Friedmann solution from the Friedmann solution, we need to perturb an infinite amount of mass and the mass perturbation remains at spatial infinity. This is also the case so as to have an asymptotically proper Friedmann solution from the decelerated Friedmann solution, for which the potential is negative. This suggests that asymptotically proper accelerated Friedmann solutions are physically acceptable as nonlinearly perturbed solutions from the Friedmann solution by some classical mechanism. This also suggests that any classical perturbation mechanism will not perturb a Friedmann universe to a quasi-Friedmann universe. Only through quantum fluctuations, it might be possible to have a quasi-Friedmann solution because an infinite amount of perturbed mass must extend in scales much larger than the Hubble horizon at any epoch. On the other hand, whether the perturbed mass is compensated or remains or even diverges at spatial infinity, the present perturbation scheme is still completely applicable for these asymptotic solutions.

D. Comparison with a perfect fluid with $p = (\gamma - 1)\rho$

For the Friedmann solution, a scalar field with exponential potential and a perfect fluid with $p = (\gamma - 1)\rho$ play a completely equivalent role. In the spatially flat case, they are related with the following relation:

$$\alpha = \frac{2}{\lambda^2} = \frac{2}{3\gamma}. \quad (5.26)$$

So, the accelerated expansion is possible if $0 < \lambda^2 < 2$ for the scalar field and if $0 < \gamma < 2/3$ for the perfect fluid. However, once we admit perturbations from a uniform distribution, the two systems get very different.

For example, a scalar wave propagates at the speed of light in the short wave length limit in the scalar field system even in the presence of potential. In contrast, in the perfect fluid system with the equation of state $p = (\gamma - 1)\rho$, a sound wave propagates at the sound speed $\sqrt{\gamma - 1}$ for $1 < \gamma \leq 2$ and, in fact, there is no sound wave but instability in the short wave length limit for $0 < \gamma < 1$ [18].

Also in the perfect fluid system with $p = (\gamma - 1)\rho$, there are two independent one-parameter families of solutions which are asymptotic to the Friedmann solution [17, 18]. One is asymptotically proper Friedmann solutions at spatial infinity and the other is asymptotically quasi-Friedmann solutions at spatial infinity. The latter is valid for both the accelerating ($0 < \gamma < 2/3$ or $\alpha > 1$) and decelerating ($2/3 < \gamma < 2$ or $1/3 < \alpha < 1$) cases, while the former is only valid for the accelerating case. Hence, the situation is exactly parallel to that in the scalar field case. This is a very unexpected result because we do admit inhomogeneity when we consider asymptotic solutions.

In the perfect fluid analysis, the strongly decelerated case, $0 < \alpha < 1/3$ or $\gamma > 2$, has not been analyzed because causality is violated in such a model. In the present analysis, on the other hand, since the scalar field with negative potential is acceptable from a causal point of view, we have included this case and found interesting features that both asymptotically proper and quasi-Friedmann solutions exist and that the asymptotically proper Friedmann solutions are very different from those for the accelerated case.

For a perfect fluid with $1 \leq \gamma \leq 2$, there is no self-similar solution which has a black-hole event horizon and is asymptotic to the proper Friedmann solution at spatial infinity. However, for a perfect fluid with $0 < \gamma < 2/3$, there is a one-parameter family of asymptotically proper Friedmann solutions. In fact, the numerical integration has revealed that there is a one-parameter family of self-similar solutions among them which contain a black-hole event horizon [23]. To implement the numerical integration in that case, it is highly advantageous that the system of ordinary differential equations has no critical surface because there is no propagation of sound wave. Also in the scalar field case, one might guess that the existence of asymptotically proper Friedmann solutions suggests the existence of self-similar black-hole solutions belonging to this class. However, the scalar field system has a critical surface coinciding with a similarity horizon, where $V_z = 1$. This makes the problem complicated because this could possibly increase the number of self-similar solutions drastically as such a critical surface may admit weak discontinuity. In this connection, we should also note that because of the critical surface, the power-law flat Friedmann solution containing a scalar field with potential is unstable for $4 < \lambda^2 < 6$ against weak discontinuity, i.e., the kink mode at a particle horizon [31]. It is however stable for $0 < \lambda^2 < 4$, marginally stable for $\lambda^2 = 4$. This kink instability might be related to the physical relevance of self-similar solutions.

VI. SUMMARY

We have considered self-similar nonlinear perturbation from the Friedmann solution and investigated the asymptotic properties of spherically symmetric self-similar solutions containing a scalar field with potential and approaching the flat Friedmann solution at spatial infinity. The potential is restricted from self-similarity to be exponential with the steepness parameter λ . This is motivated by the fact that a scalar field with sufficiently flat ($0 < \lambda^2 < 2$) potential enables the universe to expand with acceleration and hence acts as quintessence.

If the potential is so flat, i.e., $0 < \lambda^2 < 2$ that the Friedmann universe expands with acceleration, we have found that there is a one-parameter family of self-similar solutions which are asymptotic to the proper Friedmann solution at spatial infinity. Furthermore, we have found that there is also a one-parameter family of self-similar solutions which are asymptotic to the Friedmann solution but with some anomaly in solid angle. Such solutions are called asymptotically quasi-Friedmann solutions.

If the potential is steep, i.e. $\lambda^2 > 2$, we have the Friedmann universe decelerated. Even in such a potential, we have found a one-parameter family of self-similar solutions which are asymptotically quasi-Friedmann solutions. However, we have also shown that there is no nontrivial asymptotically proper Friedmann self-similar solution in this case as long as the potential is positive. We should note that it was already shown that there is no self-similar solution which contains a black-hole event horizon and is asymptotically proper Friedmann or quasi-Friedmann for a scalar field with positive potential inducing the decelerating expansion [16].

Our analysis includes the case of a massless scalar field, where the flat Friedmann universe is decelerated. In this case, we have found that there is a one-parameter family of asymptotically quasi-Friedmann self-similar solutions, while there is no nontrivial asymptotically proper Friedmann self-similar solution. We should also note that it was already shown that there is no self-similar solution which contains a black-hole event horizon and is asymptotically proper Friedmann or quasi-Friedmann for a massless scalar field [16].

Our analysis also includes the case where the potential is negative. In such a case, the Friedmann universe is strongly decelerated. We have found that there are both one-parameter families of asymptotically quasi-Friedmann self-similar solutions and asymptotically proper Friedmann self-similar solutions. The latter is very different in density and mass perturbations from that for the positive potential.

We have shown that the perturbed mass is finite for asymptotically proper Friedmann solutions as long as the potential is positive. In contrast, it is infinite for asymptotically quasi-Friedmann solutions. This suggests that asymptotically proper Friedmann solutions are physically more acceptable as solutions perturbed from the Friedmann universe through some causal mechanism than asymptotically quasi-Friedmann solutions. Although asymptotically proper Friedmann solutions are possible even if the potential is negative, the perturbed mass is infinite there.

Although we have found the above interesting properties of self-similar solutions containing a scalar field with potential, it is still an open question whether there is a self-similar black-hole solution which is asymptotically proper or quasi-Friedmann. We need possibly a numerical analysis based on the present asymptotic analysis, as it has revealed the existence of self-similar black-hole solutions for a perfect fluid with the equation of state $p = (\gamma - 1)\rho$ ($0 < \gamma < 2/3$) [18, 23]. Although spherically symmetric self-similar solutions with scalar fields have been also investigated in a dynamical systems approach [32, 33, 34], no definite answer to the existence of black-hole solutions has been reported yet. It is an important future work to answer whether there is a self-similar black-hole solution in quintessential cosmology and, if it exists, to study the physical properties of such a black-hole solution.

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